United Kingdom
Mathematics Trust

Intermediate Mathematical Challenge

Solutions 2022

For reasons of space, these solutions are necessarily brief.

There are more in-depth, extended solutions available on the UKMT website, which include some exercises for further investigation:

www.ukmt.org.uk

1. B In hours, 6 minutes equals $\frac{6}{60} = \frac{1}{10} = 0.1$.

2. A The amount of topping required is $2\frac{1}{2} \times (100 + 50 + 50)\,\text{g} = 2\frac{1}{2} \times 200\,\text{g} = 500\,\text{g} = 0.5\,\text{kg}$.

3. B The average number of eggs per nest is $\frac{3\,000\,000}{20\,000} = \frac{300}{2} = 150$.

4. D The number of days required to dig the tunnel is $\frac{2000}{5} = 400$. This is just over one year.

5. C The correct answer may be found by calculating the value of $10\,006 - 8008$ and comparing it with the values of the five options. However, it is clear that $10\,000 - 8002 = 10\,006 - 8008$ as each number on the left of the equation is 6 less than the corresponding number on the right of the equation.

6. D 20% of $3\frac{3}{4} = \frac{1}{5} \times \frac{15}{4} = \frac{3}{4}$.

7. E Let the output from the function machine be n.
Then the input is $((n + 10) \times 3 + 10) \times 3 = 3(3n + 30 + 10) = 9n + 120$.
The smallest positive integer input occurs when $n = 1$. Therefore the smallest possible positive integer which Iris could have input so that the output is also a positive integer is $9 \times 1 + 120 = 129$.

8. A 40% of 50% of 60 equals 40% of 30 = 12. Also, 50% of 60% of 70 is 50% of 42 = 21.
So the required difference is $21 - 12 = 9$.

9. E As x is greater than 2022, then $\frac{x}{2022}$ and $\frac{x+1}{2022}$ are greater than 1, whereas $\frac{2022}{x}$ and $\frac{2022}{x+1}$ are less than 1.
So we can eliminate A and C. As the remaining three fractions all have the same numerator and positive denominators, the smallest fraction is that with the largest denominator, namely $\frac{2022}{x+1}$.

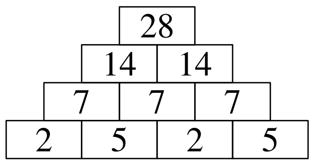
10. C The perimeter of each rectangle is $2 \times (3 + 1)$ cm $= 8$ cm.
The total perimeter, in cm, of the separate rectangles is therefore $100 \times 8 = 800$. When two rectangles are placed together at a "join" in the pattern, there is an overlap of length 1 cm and the total perimeter is reduced by 2 cm. In the complete pattern there are 99 joins and hence its perimeter, in cm, is $800 - 99 \times 2 = 800 - 198 = 602$.

11. E Let the correct answer be x. Then $\dfrac{x}{4} = 9 + x$. So $\dfrac{3x}{4} = -9$. Therefore $x = -\dfrac{4 \times 9}{3} = -12$.

12. A Let the length of the hypotenuse of the smallest triangle be l. Then $l^2 = 2^2 + 2^2 = 8$.
So $l = \sqrt{8} = 2\sqrt{2}$. Therefore the ratio of the lengths of the sides of the middle triangle to the lengths of the sides of the smallest triangle is $\sqrt{2} : 1$. Hence the corresponding ratio of the areas of these two triangles is $2 : 1$. Similarly, the area of the largest triangle is twice the area of the middle triangle. Now the area of the smallest triangle is $\dfrac{1}{2} \times 2 \times 2 = 2$. Therefore, the area of the middle triangle is $2 \times 2 = 4$. Hence the area of the large triangle is $2 \times 4 = 8$.
So the area of the shape is $2 + 4 + 8 = 14$.

13. C Let the three consecutive integers be $n - 1$, n and $n + 1$. Then $(n - 1)n(n + 1) = n - 1 + n + n + 1$. So $n(n^2 - 1) = 3n$. Hence $n(n^2 - 4) = 0$. Therefore $n = 0$ or $n^2 = 4$. So $n = -2, 0$ or 2 and the sets of three consecutive integers in which the sum of the integers equals their product are $\{-3, -2, -1\}$, $\{-1, 0, 1\}$ and $\{1, 2, 3\}$.

14. D Let each number in the second row of the number pyramid be n. Then the numbers in the third row are both $2n$ and the top number is $4n$. So the top number is a multiple of 4.
Hence option D is correct and therefore option E is incorrect.

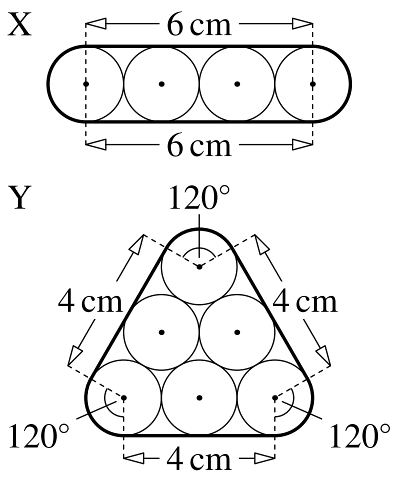


Also, the number pyramid above provides counterexamples to options A, B and C.

15. E The line l is the perpendicular bisector of the line segment joining points $(5, 3)$ and $(1, -1)$.
The midpoint of this line segment is $\left(\dfrac{5 + 1}{2}, \dfrac{3 + (-1)}{2}\right) = (3, 1)$.
The gradient of the line segment is $\dfrac{3 - (-1)}{5 - 1} = \dfrac{4}{4} = 1$. Hence the gradient of l is -1.
Let the equation of l be $y = -x + c$. As l passes through the point $(3, 1)$, $1 = -3 + c$. So $c = 4$.
Hence the equation of l is $y = -x + 4$ or $y = 4 - x$.

16. D Note that $4^{2022} = (2^2)^{2022} = 2^{4044}$. Therefore half of 4^{2022} is $2^{4044} \div 2 = 2^{4043}$.

17. **C** As shown in the upper figure, X consists of two straight portions of length 6 cm and two semicircular arcs of radius 1 cm.
So its length is $(2 \times 6 + 2 \times \frac{1}{2} \times 2\pi \times 1)$ cm $= (12 + 2\pi)$ cm.



In the lower figure, the band Y turns through one third of a revolution at each corner, so it consists of three straight portions of length 4 cm and three arcs, each of which is one third of the circumference of a circle of radius 1 cm.
Hence its length is $(3 \times 4 + 3 \times \frac{1}{3} \times 2\pi \times 1)$ cm
$= (12 + 2\pi)$ cm.

So the two bands have the same length.

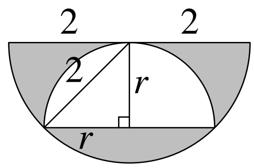
18. **B** Let the cost of Slack Bess be £C. Then the profit on the horse is $C\%$.
Therefore $C(1 + \frac{C}{100}) = 56$. So $C^2 + 100C - 5600 = 0$. Hence $(C + 140)(C - 40) = 0$.
Therefore, as $C > 0, C = 40$. So Slack Bess cost £40.

19. **A** The circumference of the circle is $2\pi \times 6 = 12\pi$.

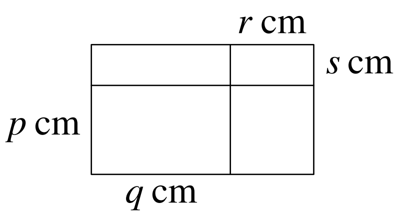
So the area of the sector shown is $\dfrac{10}{12\pi} \times \pi \times 6^2 = \dfrac{10 \times 36}{12} = 30$.

20. **D** Let the mean of Aroon's five integers be m. Then the median is $m + 2$ and the mode is $m + 4$.
Hence we can let the five integers in ascending order be $x, y, m + 2, m + 4, m + 4$, where x and y are to be determined. Since the mean of the integers is m, their sum is $5m$.
So $x + y + 3m + 10 = 5m$. Therefore $x + y = 2m - 10$. Now for the range of the integers to be as large as possible, x needs to be as small as possible. In turn, this means that y must be as large as possible. We know that y is an integer which is less than $m + 2$, so its maximum value is $m + 1$.
So the smallest possible value of x is $2m - 10 - (m + 1) = m - 11$.
Hence the largest possible value of the range of the integers is $m + 4 - (m - 11) = 15$.

21. **B** First note that the radius of the larger semicircle is 2.
So its area is $\frac{1}{2} \times \pi \times 2^2 = 2\pi$.
Let the radius of the smaller semicircle be r.
Then, by Pythagoras' Theorem, $r^2 + r^2 = 2^2$. So $r^2 = 2$.
Therefore the area of the smaller semicircle is $\frac{1}{2} \times \pi \times 2 = \pi$.
So the shaded area is $2\pi - \pi = \pi$.
Hence half the area of the large semicircle is shaded.



22. **D** Let the largest rectangle measure p cm by q cm and the smallest rectangle measure r cm by s cm, as shown.
Then $2p + 2q = 28$ and $2r + 2s = 12$.
Now the perimeter, in cm, of the original rectangle is $2p + 2q + 2r + 2s = 28 + 12 = 40$.
Hence $p + q + r + s = (p + s) + (q + r) = 20$.

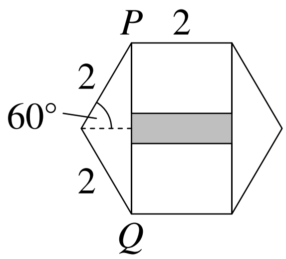


So the length and width, in cm, of the original rectangle sum to 20.
The area of the original rectangle is the product of its width and length. Of options given, only an area of 96 cm^2 gives a possible width and length which sum to 20 cm, namely 8 cm and 12 cm.
(*It is left to the reader to show that none of the other four options corresponds to integer values for the width and length which sum to* 20 cm.)

23. **E** The sum of the interior angles of a hexagon is $(6 - 2) \times 180° = 720°$.
Hence each interior angle of a regular hexagon is $720° \div 6 = 120°$.

In the figure, $\sin 60° = \dfrac{PQ/2}{2}$. So $PQ = 4\sin 60° = 4 \times \dfrac{\sqrt{3}}{2} = 2\sqrt{3}$.

Therefore the height of the area of overlap is $2 + 2 - 2\sqrt{3} = 4 - 2\sqrt{3}$.
The length of the area of overlap is 2.
Hence the area of overlap of the two squares is $2(4 - 2\sqrt{3}) = 8 - 4\sqrt{3}$.



24. **B** Let the costs, in pounds, of apple, blueberry, cherry and damson pies be a, b, c, d respectively.
Then $c = 2a$, $b = 2d$ and $c + 2d = a + 2b$.
Substituting for b and c in the third equation gives $2a + 2d = a + 4d$. Therefore $a = 2d$.
So $c = 2 \times 2d = 4d$.
Now the cost, in pounds, of one of each type of pie is $a + b + c + d = 2d + 2d + 4d + d = 9d$.
So the amount in pounds which Paul spends is a multiple of 9.
Of the given options, only £18 satisfies this requirement.

25. **C** Let the stones measure 1 unit by 1 unit.
Then, after the border is added, the patio measures $x + 2$ by $y + 2$.
So the area of the border is $(x + 2)(y + 2) - xy$.
Therefore $(x + 2)(y + 2) - xy = xy$. So $xy + 2x + 2y + 4 - xy = xy$, that is $xy - 2x - 2y = 4$.
Hence $xy - 2x - 2y + 4 = 8$, that is $(x - 2)(y - 2) = 8$.
The table shows possible integer values of $x - 2$ and the corresponding values of $y - 2$, x and y.

$x - 2$	$y - 2$	x	y
−8	−1	−6	1
−4	−2	−2	0
−2	−4	0	−2
−1	−8	1	−6
1	8	3	10
2	4	4	6
4	2	6	4
8	1	10	3

Note that both x and y must be positive, so there are four possible values of x, namely 3, 4, 6, 10.